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OPTIMAL KINEMATICS FOR FINITE ELEMENTS WITH SMEARED-EMBEDDED DISCONTINUITY

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Summary. In this paper an optimal kinematics for finite elements with smeared non-uniform discontinuity/crack is proposed to eliminate the spurious stress transfer across a fully softened crack (i.e. stress locking). We first present the optimal kinematics of the finite elements with embedded non-uniform displacement jumps. It is found that, if the regularization bandwidth of the strong discontinuity reaches a critical value, i.e. the so-called consistent characteristic length of an element, the concept of classical smeared crack model is recovered. The optimal definition of the smeared cracking (inelastic) strain is then established such that the stress locking is completely removed. Finally, a constant stress triangle with a non-uniform discontinuity is analytically solved. The prediction shows that finite elements with the proposed kinematics can capture the expected stress and strain states even if a smeared crack model is used.

1 INTRODUCTION

When applied to the numerical modeling of concrete cracking/fracture, both the classical smeared and discrete crack models generally exhibit the spurious stress transfer across a fully softened crack (i.e. the so-called stress locking), unless the crack propagation path is parallel to the edge of finite element meshes. To solve this problem the more advanced but more complex finite element methods enhanced with the local (element-wisely embedded) and the global (the partition of unity based) strong discontinuities have been introduced and widely investigated in the last two decades. However, these advanced methods require burdensome modifications to the existing programming codes based on the continuum mechanics and smeared crack models.

The careful inspection revealed that after the crack/discontinuity is initiated the poor representation of the kinematics in classical smeared crack models is responsible for the stress locking. As an effort to remedy the aforementioned problems, in this paper we propose an optimal kinematics for the finite elements with smeared non-uniform discontinuity/crack.

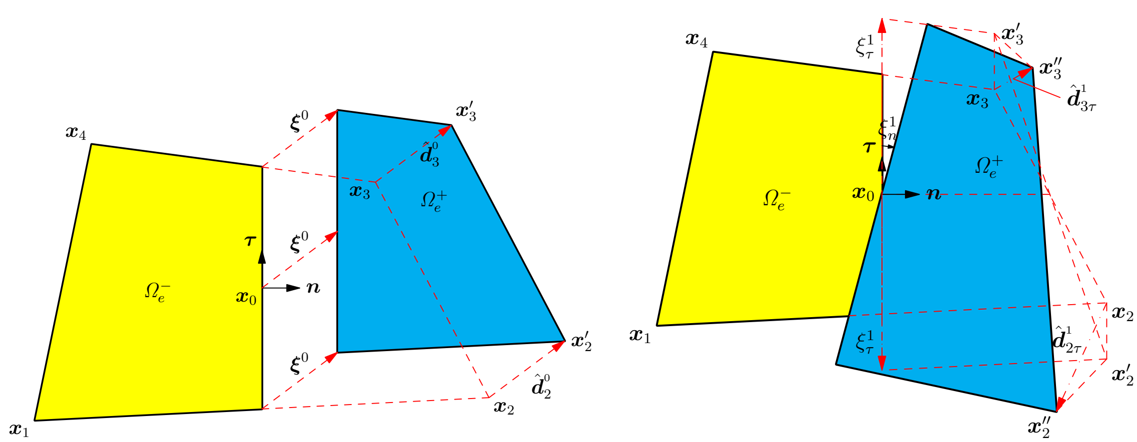
2 OPTIMAL REPRESENTATION OF KINEMATICS

2.1 Kinematics of embedded strong discontinuity

Upon the assumption of infinitesimal deformation, assume that a 2-D finite element Ω_e with surface area A_e is divided into two sub-domains Ω_e^+ and Ω_e^- by a straight discontinuity $\mathscr{S}$ with its length denoted by $l_{\mathscr{S}}$. The normal and tangent unit vectors of the discontinuity segment are represented by $\boldsymbol{n}$ (pointing towards Ω_e^+) and $\boldsymbol{\tau}$, respectively. An associated local coordinate system s is defined along the discontinuity $\mathscr{S}$ measured from its centroid $\boldsymbol{x}_0$ such that $s \in [-l_{\mathscr{S}}/2, l_{\mathscr{S}}/2]$. We consider non-unform displacement jumps $[\![\boldsymbol{u}]\!]$ across the discontinuity $\mathscr{S}$, and denote its constant and linear parts by $\boldsymbol{\xi}^0$ and $\boldsymbol{\xi}^1$, respectively [1]. Note that the displacement jump parameters $\boldsymbol{\xi} = \{\boldsymbol{\xi}^0, \boldsymbol{\xi}^1\}^{\mathrm{T}}$ are defined at the local (element) level. Therefore, the displacement field $\boldsymbol{u}$ in the solid Ω_e can be expressed as

$$\boldsymbol{u}(\boldsymbol{x}) = \tilde{\boldsymbol{u}}(\boldsymbol{x}) + \big(H_{\mathscr{S}} - \varphi\big)\hat{\boldsymbol{u}}(\boldsymbol{x}), \qquad \varphi = \sum N_i \qquad (i \in \Omega_e^+) \tag{1}$$

where $\tilde{\boldsymbol{u}}$ is the regular continuous displacement interpolated through a standard set of shape function $\boldsymbol{N}$ and the associated nodal displacements $\boldsymbol{d}$, i.e. $\tilde{\boldsymbol{u}} \approx \boldsymbol{N}\boldsymbol{d}$; $\hat{\boldsymbol{u}}$ denotes the discontinuous displacement field due to the rigid body motion; $H_{\mathscr{S}}$ is the Heaviside function defined as $H_{\mathscr{S}}(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in \Omega_e^+$ and $H_{\mathscr{S}}(\boldsymbol{x}) = 0$ otherwise; φ is a continuous function defined in terms of N_i associated with the set of nodes $i \in \Omega_e^+$ [2].



(a) Constant discontinuity mode

(b) Linear discontinuity mode

Figure 1: Sketch of the constant and linear discontinuity modes.

For finite elements with constant or linear discontinuity illustrated in Fig.1, the discontinuous displacement field $\hat{\boldsymbol{u}}$ at point $\boldsymbol{x}$ and its symmetric gradient can be expressed as

$$\hat{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{\xi}^0 + \Big[\xi_n^1\big(\boldsymbol{n}\otimes\boldsymbol{\tau} - \boldsymbol{\tau}\otimes\boldsymbol{n}\big) + \xi_\tau^1 \boldsymbol{\tau}\otimes\boldsymbol{\tau}\Big]\hat{\boldsymbol{x}}, \qquad \nabla^{\mathrm{s}}\hat{\boldsymbol{u}} = \xi_\tau^1\big(\boldsymbol{\tau}\otimes\boldsymbol{\tau}\big) \tag{2}$$

with $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_0$, where $\xi_n^1 = \boldsymbol{\xi}^1 \cdot \boldsymbol{n}$ and $\xi_\tau^1 = \boldsymbol{\xi}^1 \cdot \boldsymbol{\tau}$ signify the normal and tangential components of the linear discontinuity parameter $\boldsymbol{\xi}^1$, respectively. The compatible strain $\boldsymbol{\epsilon}$ in the solid is then evaluated as the additive form of its continuous part $\boldsymbol{\epsilon}^{\mathrm{co}}$ and discontinuous part $\boldsymbol{\epsilon}^{\mathrm{cr}}$

$$\boldsymbol{\epsilon} = \nabla^{\mathrm{s}}\boldsymbol{u} = \boldsymbol{\epsilon}^{\mathrm{co}} + \boldsymbol{\epsilon}^{\mathrm{cr}}, \quad \boldsymbol{\epsilon}^{\mathrm{co}} = \nabla^{\mathrm{s}}\tilde{\boldsymbol{u}} + \big(H_{\mathscr{S}} - \varphi\big)\nabla^{\mathrm{s}}\hat{\boldsymbol{u}} - \big(\hat{\boldsymbol{u}}\otimes\nabla\varphi\big)^{\mathrm{s}}, \quad \boldsymbol{\epsilon}^{\mathrm{cr}} = \delta_{\mathscr{S}}\big(\hat{\boldsymbol{u}}\otimes\boldsymbol{n}\big)^{\mathrm{s}} \tag{3}$$

where $\delta_{\mathscr{S}}$ is the Dirac-delta function along the discontinuity $\mathscr{S}$. When $A_e/l_{\mathscr{S}}$ approaches to zero [3], the support of function φ will disappear and then the contribution from the term $-\varphi\nabla^{\mathrm{s}}\hat{\boldsymbol{u}}$ can be neglected, such that

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{co}} + \boldsymbol{\epsilon}^{\mathrm{cr}}, \qquad \boldsymbol{\epsilon}^{\mathrm{co}} = \nabla^{\mathrm{s}}\tilde{\boldsymbol{u}} + H_{\mathscr{S}}\nabla^{\mathrm{s}}\hat{\boldsymbol{u}} - \big(\hat{\boldsymbol{u}} \otimes \nabla\varphi\big)^{\mathrm{s}}, \qquad \boldsymbol{\epsilon}^{\mathrm{cr}} = \delta_{\mathscr{S}}\big(\hat{\boldsymbol{u}} \otimes \boldsymbol{n}\big)^{\mathrm{s}} \tag{4}$$

Moreover, if only constant discontinuity is considered, i.e. $\boldsymbol{\xi}^1 = \boldsymbol{0}$ which leads to $\hat{\boldsymbol{u}} = \boldsymbol{\xi}^0 = [\![\boldsymbol{u}]\!]$ and $\nabla^{\mathrm{s}}\hat{\boldsymbol{u}} = \boldsymbol{0}$, the compatible strain $\boldsymbol{\epsilon}$ in the solid then reduces to

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{co}} + \boldsymbol{\epsilon}^{\mathrm{cr}}, \qquad \boldsymbol{\epsilon}^{\mathrm{co}} = \nabla^{\mathrm{s}}\tilde{\boldsymbol{u}} - \big([\![\boldsymbol{u}]\!] \otimes \nabla\varphi\big)^{\mathrm{s}}, \qquad \boldsymbol{\epsilon}^{\mathrm{cr}} = \delta_{\mathscr{S}}\big([\![\boldsymbol{u}]\!] \otimes \boldsymbol{n}\big)^{\mathrm{s}} \tag{5}$$

Eqs.(4) and (5) are the kinematics used in the strong discontinuity approach [1, 2].

2.2 Kinematics of smeared discontinuity/crack

In numerical implementation the unbounded Dirac-delta function $\delta_{\mathscr{S}}$ is generally regularized as $\delta_{\mathscr{S}} = \Xi_{\mathscr{S}}/k$, where k is the regularization bandwidth; $\Xi_{\mathscr{S}}$ is a collocation function placed in $\mathscr{S}$, i.e. $\Xi_{\mathscr{S}}(\boldsymbol{x}) = 1\ \forall\boldsymbol{x} \in \mathscr{S}$ and $\Xi_{\mathscr{S}} = 0$ otherwise. When the bandwidth k tends to zero the kinematic state of strong discontinuity is recovered. If the bandwidth $0 < k < l_{\mathrm{ch}}$ with l_{ch} being a critical length of the finite element, one obtains the so-called weak discontinuity .

In this section we consider the case $k = l_{\mathrm{ch}}$ which means that the discontinuous displacement field $\hat{\boldsymbol{u}}$ in Eq.(2) is smeared over the entire area of the element, i.e. $\Xi_{\mathscr{S}}(\boldsymbol{x}) = 1\ \forall\boldsymbol{x} \in \Omega_e$. Under such case the strain $\boldsymbol{\epsilon}$ in the solid can be written into the following additive form

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{co}} + \boldsymbol{\epsilon}^{\mathrm{cr}}, \qquad \boldsymbol{\epsilon}^{\mathrm{cr}} = \frac{1}{l_{\mathrm{ch}}}\big(\hat{\boldsymbol{u}} \otimes \boldsymbol{n}\big)^{\mathrm{s}} \tag{6}$$

In the concept of smeared crack the continuous condition $\boldsymbol{u} = \tilde{\boldsymbol{u}}$ holds, i.e. In Eqs.(1) and (3) the terms related to $H_{\mathscr{S}} - \nabla\varphi = 0$ disappear, and we have

$$\frac{1}{l_{\mathrm{ch}}}\big(\hat{\boldsymbol{u}} \otimes \boldsymbol{n}\big)^{\mathrm{s}} = \big(\hat{\boldsymbol{u}} \otimes \nabla\varphi\big)^{\mathrm{s}} \quad \Longrightarrow \qquad l_{\mathrm{ch}} = \big(\nabla\varphi \cdot \boldsymbol{n}\big)^{-1}, \qquad \boldsymbol{\epsilon}^{\mathrm{cr}} = \big(\hat{\boldsymbol{u}} \otimes \nabla\varphi\big)^{\mathrm{s}} \tag{7}$$

Note that, the definition of the critical length of an element is exactly the same as the consistent characteristic length derived in [4] where the equivalence of energy dissipation was assumed between the smeared and embedded discontinuity models. Moreover, for a linear 3-node constant strain triangular (CST) if only constant discontinuity is considered, Eq.(7) reduces to the inelastic strain defined in the smeared-embedded crack model [5].

3 EXEMPLARY ANALYSIS

In this section we will present exemplary analysis of an asymmetric CST subjected to tension [6]. Due to the asymmetry, the vertical discontinuity, introduced at the centroid, opens at one of the edges only, as depicted in Fig.(2), where w_x is the crack opening (i.e. $\xi_\tau^0 = \xi_\tau^1 = 0$).

Following somewhat complex but straightforward mathematic manipulations, the stress field inside the CST is given, in matrix form, by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \big[\mathbb{E}_0\big]\Big(\{\boldsymbol{\epsilon}\} - \{\boldsymbol{\epsilon}^{\mathrm{cr}}\}\Big) = \frac{E_0}{1-\nu_0^2}\begin{bmatrix} 1 & \nu_0 & 0 \\ \nu_0 & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu_0}{2} \end{bmatrix}\begin{Bmatrix} 1 \\ -\nu_0 \\ 0 \end{Bmatrix}\epsilon_{xx}^{\mathrm{co}} = \begin{Bmatrix} E_0\epsilon_{xx}^{\mathrm{co}} \\ 0 \\ 0 \end{Bmatrix} \tag{8}$$

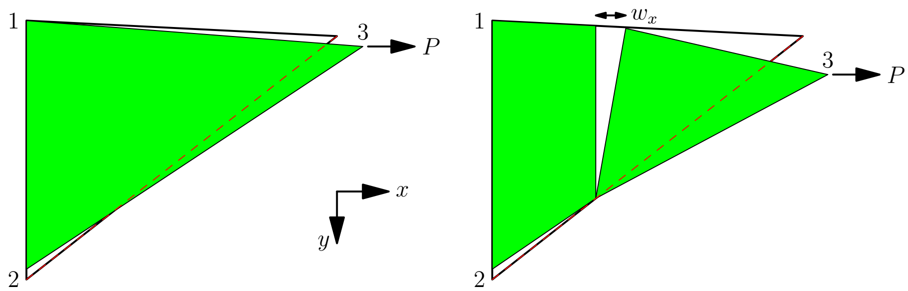


Figure 2: 3-node plane stress CST element subjected to uniaxial tension.

where $\epsilon_{xx}^{\mathrm{co}} = \sigma_{xx}/E_0$ denotes the continuum strain along the loading axis x, with σ_{xx} being the tensile stress along axis x; $\left[\mathbb{E}_0\right]$ is the elastic stiffness matrix in terms of Young's modulus E_0 and Poisson's ratio ν_0; the cracking strain is expressed as $\left\{\boldsymbol{\epsilon}^{\mathrm{cr}}\right\} = \boldsymbol{B}_3 \cdot \hat{\boldsymbol{u}}_3$ with $\boldsymbol{B}_3$ representing the displacement–strain matrix corresponding to the solitary node 3.

As can be seen from Eq.(8), except that the x-axial stress component σ_{xx} will transfer stress which tends to zero as crack opening w_x approaches to infinity, i.e. $\epsilon_{xx}^{\mathrm{co}} = \epsilon_{xx} - \epsilon_{xx}^{\mathrm{cr}} \to 0$ when $\epsilon_{xx}^{\mathrm{co}} \to \epsilon_{xx}$, the other stress components remain zero all along.

4 CONCLUSIONS

It seems that, if the kinematics due to the rigid body motions can be fully represented and a consistent characteristic length is used to regularize the softening regimes of stress–strain curve, finite elements with smeared crack/disconuituity are free of stress locking and can still be used in the numerical modeling of concrete cracking/fracture.

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